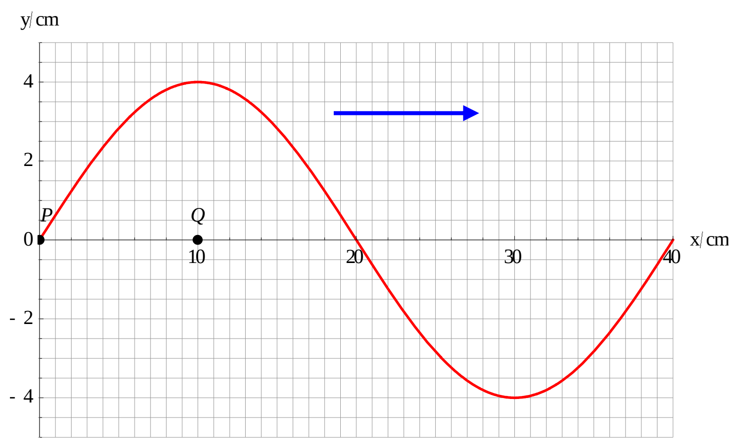
Teacher notes
Topic C

A longitudinal wave puzzle

Suppose a longitudinal wave travels through a medium from left to right. Consider the equilibrium positions of two particles in the medium: P at the origin and Q a distance *q* away, to the right of P. What are the maximum and minimum distances between P and Q as time goes on?

So consider the wave, at *t* = 0, shown in the figure (here *q* = 10 cm):



P and Q are the **equilibrium** positions of the two particles in the medium. This means that when there is no wave in the medium the positions of P and Q are 0 cm and 10 cm respectively. When a wave is present the positions of P and Q will change according to the red curve in the graph. So, at *t* = 0, P has displacement 0 and so its position remains at 0. But Q has displacement 4 cm so its position is at 10 + 4 = 14 cm. The distance between P and Q at *t* = 0 is then 14 cm.

At any time, *t*, the distance between P and Q is given by $D = q + y_Q - y_P$, where y_P and y_Q are the displacements of P and Q. Now, $y_P = -A\sin(\omega t)$ and $y_Q = A\cos(\omega t)$ (*A* is the amplitude) so that:

$$D = q + A\cos(\omega t) + A\sin(\omega t)$$

Try to guess the maximum distance between P and Q before reading on.

Since *q* = 10 cm and *A* = 4 cm the distance between P and Q is

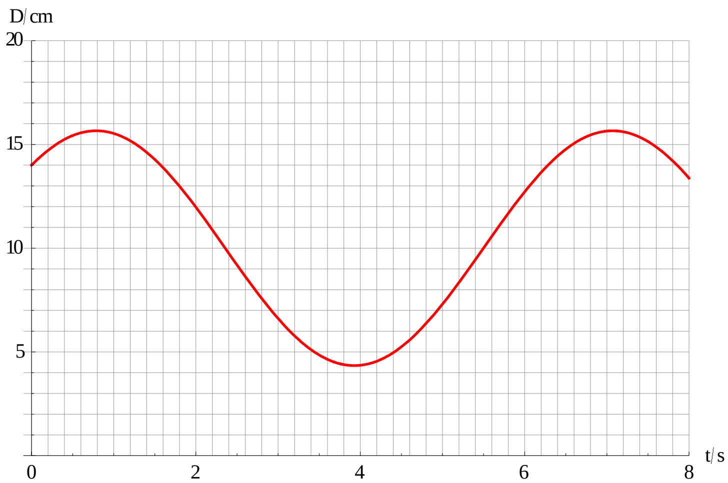
$$D = 10 + 4\cos(\omega t) + 4\sin(\omega t)$$

We are interested in finding the maximum value of this distance. With trigonometry we find:

$$D = 10 + 4\sqrt{2}\sin(\omega t + \frac{\pi}{4})$$

The maximum value is when the sine is equal to 1 i.e. $D_{max} = 10 + 4\sqrt{2} \approx 15.7$ cm. The minimum distance is $D_{min} = 10 - 4\sqrt{2} \approx 4.34$ cm.

We get these answers also by plotting the distance *D*:



The initial distance between P and Q is 14 cm. This makes sense: the displacement of P at *t* = 0 is zero so its position is at *x* = 0. The displacement of Q is 4 cm, so it displaced 4 cm to the right of its equilibrium position at *x* = 10 i.e. finds itself at *x* = 14 cm. The maximum is at $\omega t = \frac{\pi}{4} \approx 0.785$ and the minimum occurs at $\omega t = \frac{5\pi}{4} \approx 3.93$.

The graph above was made with $\omega = 1$ rad s^{-1}. Changing ω does not make a difference to the conclusion.

Figure 1 shows the wave when P and Q have their maximum separation:

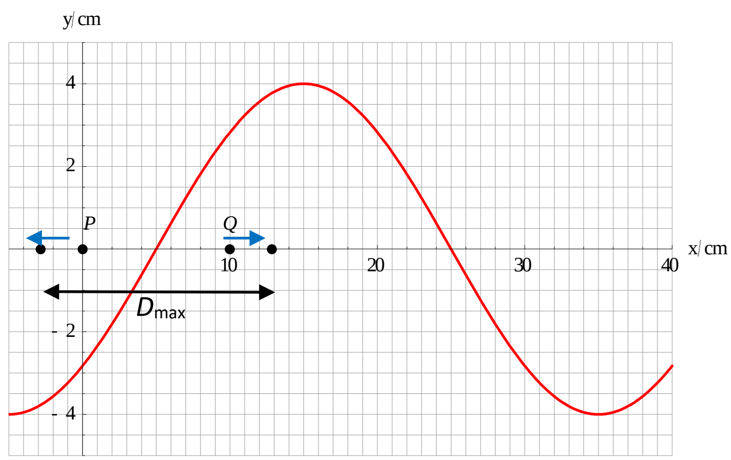


Figure 1

P is at position $x = 0 - 2.83 = -2.83$ cm and Q at $x = 10 + 2.83 = +12.83$ cm for a separation of 15.7 cm as we found before. Finally, shown below is the wave when P and Q are at their closest, Figure 2.

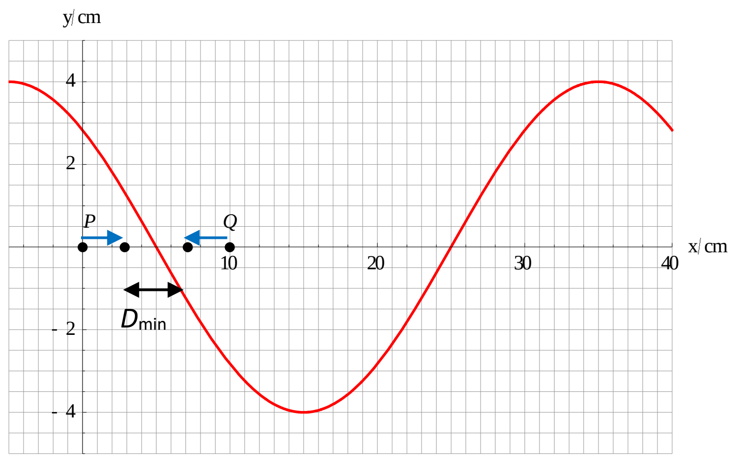
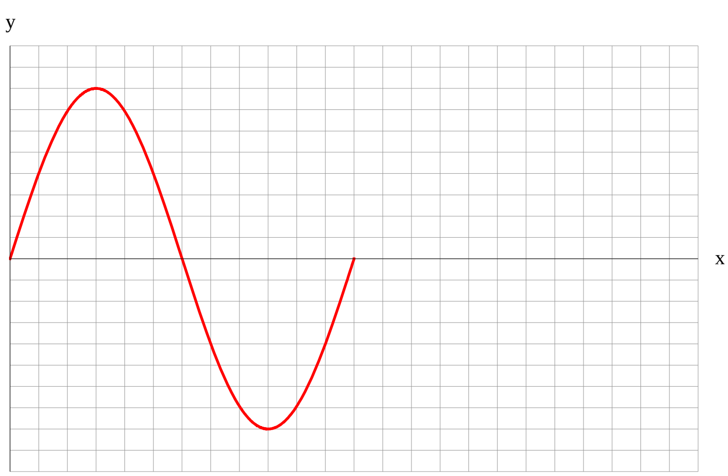


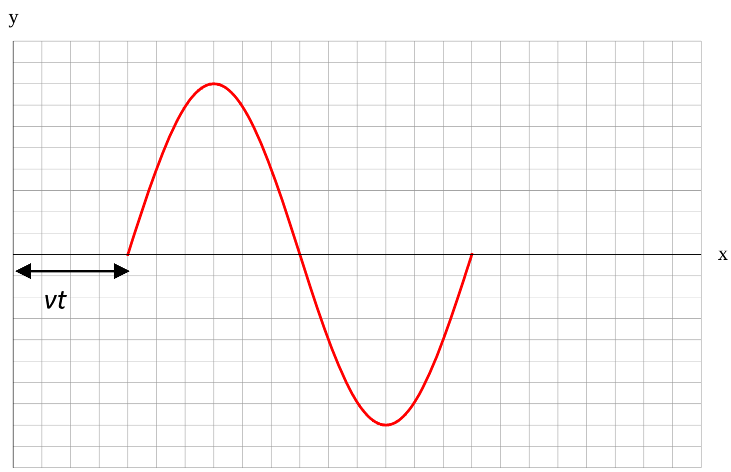
Figure 2

To justify the graphs in Figures 1 and 2 we need an alternative, more advanced approach. We need to derive the general equation of a wave as a function of both position and time.

Shown is the graph of the function $y = A\sin(\frac{2\pi}{\lambda}x)$ which represents the displacement of a wave as a function of the distance *x* at *t* = 0. (We show only one full wave for clarity.)



After time *t* the wave will move forward a distance *vt* and so the wave will look like:



We have shifted the original graph by *vt* to the right and from math (function transformations) we know that the equation of this graph will be $f(x) \to f(x - vt)$. So $y = A\sin(\frac{2\pi}{\lambda}x)$ becomes

$$\begin{aligned} y &= A\sin(\frac{2\pi}{\lambda}(x - vt)) \\ &= A\sin(\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}vt) \\ &= A\sin(\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}\frac{\lambda}{T}t) \\ &= A\sin(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t) \\ y &= A\sin(\frac{2\pi}{\lambda}x - \omega t) \end{aligned}$$

This is the equation of a wave travelling to the right.

Thus, the equation of the graph of Figure 1 is $y = A\sin(\frac{2\pi}{\lambda}x - \frac{\pi}{4}) = 4\sin(\frac{\pi x}{20} - \frac{\pi}{4})$ and that of Figure 2 is $y = A\sin(\frac{2\pi}{\lambda}x - \frac{5\pi}{4}) = 4\sin(\frac{\pi x}{20} - \frac{5\pi}{4})$.

Incidentally, we can use the general equation to write down the displacements of P and Q at an arbitrary time that we used earlier without much justification: for P, $x = 0$ so $y_P = 4\sin(0 - \omega t) = -4\sin(\omega t)$ and for Q, $y = 4\sin(\frac{\pi \times 10}{20} - \omega t) = 4\sin(\frac{\pi}{2} - \omega t) = 4\cos(\omega t)$ just as said earlier.